

# Turbulent convection: comparing the moment equations to numerical simulations.

F. Kupka<sup>1</sup>

*Institute for Astronomy, University of Vienna, Türkenschanzstraße 17, A-1180 Wien, Austria*  
 kupka@astro.univie.ac.at

## ABSTRACT

The non-local hydrodynamic moment equations for compressible convection are compared to numerical simulations. Convective and radiative flux typically deviate less than 20% from the 3D simulations, while mean thermodynamic quantities are accurate to at least 2% for the cases we have investigated. The moment equations are solved in minutes rather than days on standard workstations. We conclude that this convection model has the potential to considerably improve the modelling of convection zones in stellar envelopes and cores, in particular of A and F stars.

*Subject headings:* hydrodynamics — turbulence — convection — stars: interiors — stars: atmospheres

## 1. Introduction

In a wide range of astrophysical problems heat is transported by both radiation and convection. Examples include stellar envelopes and stellar cores, where convection may be coupled with rotation, pulsation, and diffusion. To be useful, a convection model must be both manageable and reliable. Local convection models satisfy both criteria but they are restricted to regions of strong to moderately strong convection, while in most cases one needs to model moderately strong to weak convection.

Numerical simulations (Sofa and Chan (1984); Freytag et al. (1996), and Atroshchenko and Gadun (1994); Kim and Chan (1998); Stein and Nordlund (1998)) have been applied to 2D and 3D stellar atmospheres. Their successful reproduction of spectral line profiles and solar granulation statistics has proved their reliability. However, they are restricted to layers near the stellar surface (Kupka 1999a) because of the huge thermal time scales characterizing stellar interiors. In addition, they currently are too expensive for applications such as non-linear stellar pulsation cal-

culations for RR Lyrae stars (Feuchtinger 1998) or for use in everyday spectrum synthesis of large wavelength ranges or large parameter sets for A-M stars. Hence, they do not satisfy the criterion of manageability for the whole range of problems where convection occurs.

Convection models based on the non-local, hydrodynamic moment equations provide a possible alternative. They describe the moments of an ensemble average of the basic fields: velocity  $U$ , temperature  $T$ , and density  $\rho$  (or pressure  $P$ ). Dynamic equations for the moments are derived directly from the fully compressible Navier-Stokes equations (NSE, Canuto (1997)). The “ensemble” used in this averaging process consists of realizations of solutions of the NSE (Xiong 1986, 1997; Canuto 1992, 1993, 1997; Canuto and Dubovikov 1998), or of “convective elements” of different velocity and temperature (Grossman et al. 1993; Grossman 1996). However, the moment equations entail higher order moments and thus require closure assumptions. To obtain a closed set of equations most convection models have invoked a mixing length. The models derived in Canuto (1992, 1993, 1997) and in Canuto and Dubovikov (1998) avoid a mixing length by providing a fully non-local set of dynamic equations for the second order moments.

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<sup>1</sup> Institute of Mathematics, University of Vienna, Strudlhofgasse 4, A-1090 Wien, Austria

Recently, Kupka (1999a) has presented the first comparison of a variant of these convection models with numerical simulations of compressible convection for a stellar-like scenario. He used the downgradient approximation (DGA) for the third order moments. To avoid its shortcomings, here we use a more complete model. It is based on a model introduced in Canuto (1992). The latter was successfully applied to the convective boundary layer of the terrestrial atmosphere (Canuto et al. 1994). We describe the physical scenario for our comparison and present results for two sample problems. We also consider the potential of such models for application to envelope convection in A and F stars.

## 2. The physical scenario

In Muthsam et al. (1995, 1999) the fully compressible NSE were solved for a 3D plane parallel geometry with a constant gravity  $g$  pointing to the bottom of a simulation box. Periodic boundary conditions were assumed horizontally. A constant temperature  $T$  was prescribed at the top and a constant input flux at the bottom of the box ( $\partial T/\partial z = \text{const.}$ ,  $z$  denotes the vertical direction, i.e. top to bottom). Top and bottom were taken to be impenetrable and stress free. A perfect gas law was assumed and radiation was treated in the diffusion approximation. Stable and unstable layers were defined through  $\partial T/\partial z$ , which initially was a piecewise linear function in units of the adiabatic gradient:  $\partial T/\partial z = b(z)(\partial T/\partial z)_{\text{ad}}$ , where  $b(z) = 1 + (\nabla - \nabla_{\text{ad}})/\nabla_{\text{ad}}$ . To define the stability properties of the layers a Rayleigh number  $Ra$  for one zone where  $b(z) = \text{const.}$  and  $> 1$  (Muthsam et al. 1995, 1999) is specified together with a Prandtl number  $Pr$  and an initial temperature contrast. This yields the radiative conductivity  $K_{\text{rad}}(z)$ . Both  $Pr$  and  $K_{\text{rad}}$  are kept fixed and place regions of stable and unstable stratification at different vertical locations in the simulation box. A similar approach was used by Cattaneo et al. (1991); Chan and Sofia (1996); Hurlburt et al. (1994); Porter and Woodward (1994); Singh et al. (1995) and others. The comparison to simulations with a prescribed viscosity avoids the need to use a subgrid scale model. According to our numerical experiments with the moment equations, molecular viscosity can decrease the efficiency of convection as measured by  $F_{\text{conv}}/F_{\text{total}}$  by up to 15%

and smooths out the numerical solution in stably stratified regions (Kupka 1999a,b).

## 3. A convection model based on the hydrodynamic moment equations

Using the Reynolds stress approach, Canuto (1992, 1993) has derived a convection model which consists of four differential equations for the basic second order moments  $K$  (turbulent kinetic energy),  $\overline{\theta^2}$  (mean square of temperature fluctuations, i.e. thermal potential energy),  $\overline{w\theta}$  ( $F_c = c_p \rho J = c_p \rho \overline{w\theta}$  is the convective flux), and the vertical turbulent kinetic energy  $\overline{w^2}$ . The model was rederived by Canuto and Dubovikov (1998, CD98) using a new turbulence model based on renormalization group techniques. In CD98 notation, the convection model reads ( $\partial_t \equiv \partial/\partial t$ ,  $\partial_z \equiv \partial/\partial z$ ):

$$\partial_t K + D_f(K) = g\alpha J - \epsilon + \frac{1}{2}C_{ii} + \partial_z(\nu \partial_z K), \quad (1)$$

$$\begin{aligned} \partial_t(\frac{1}{2}\overline{\theta^2}) + D_f(\frac{1}{2}\overline{\theta^2}) &= \beta J - \tau_\theta^{-1}\overline{\theta^2} \\ &+ \frac{1}{2}\partial_z(\chi \partial_z \overline{\theta^2}) + \frac{1}{2}C_\theta, \end{aligned} \quad (2)$$

$$\begin{aligned} \partial_t J + D_f(J) &= \beta \overline{w^2} + \frac{2}{3}g\alpha \overline{\theta^2} - \tau_p^{-1}J \\ &+ \frac{1}{2}\partial_z(\chi \partial_z J) + C_3 + \frac{1}{2}\partial_z(\nu \partial_z J), \end{aligned} \quad (3)$$

$$\begin{aligned} \partial_t(\frac{1}{2}\overline{w^2}) + D_f(\frac{1}{2}\overline{w^2}) &= -\tau_{\text{pv}}^{-1}(\overline{w^2} - \frac{2}{3}K) \\ &+ \frac{2}{3}g\alpha J - \frac{1}{3}\epsilon + \frac{1}{2}C_{33} + \frac{1}{2}\partial_z(\nu \partial_z \overline{w^2}) \end{aligned} \quad (4)$$

$$\begin{aligned} \partial_t \epsilon + D_f(\epsilon) &= c_1 \epsilon K^{-1} g\alpha J - c_2 \epsilon^2 K^{-1} + c_3 \epsilon \tilde{N} \\ &+ \partial_z(\nu \partial_z \epsilon), \quad \tilde{N} \equiv \sqrt{g\alpha|\beta|}, \end{aligned} \quad (5)$$

$$D_f(\epsilon) \equiv \partial_z(\overline{\epsilon w}) = -\frac{1}{2}\partial_z[(\nu_t + \chi_t)\partial_z \epsilon] \quad (6)$$

Here,  $\alpha$  is the volume compressibility ( $= 1/T$  for a perfect gas),  $\beta$  is the superadiabatic gradient,  $\chi = K_{\text{rad}}/c_p \rho$ , and  $\nu = \chi Pr$ . The  $\tau$ 's are time scales. We use (25a), (27b), and (28b) of CD98 to relate the latter to the dissipation time scale  $\tau = 2K/\epsilon$ . Compressibility effects are represented by  $C_{ii}, C_\theta, C_3, C_{33}$  given by equations (42)–(48)

of Canuto (1993). We neglected a few terms of  $C_{ii}, C_\theta, C_3, C_{33}$  that were too small to contribute to the solution of the moment equations (Kupka 1999b). In equation (6),  $\nu_t = C_\mu K^2/\epsilon$ , and  $C_\mu$  is a constant given by (24d) of CD98, for which we take the Kolmogorov constant  $Ko = 1.70$ , while  $\chi_t$  is given by the low viscosity limit of (11f) of CD98. We optionally included molecular dissipation by restoring the largest (i.e. second order moment) terms containing the kinematic viscosity  $\nu$  (i.e.  $\partial_z(\nu\partial_z K)$ , etc.). They are important when  $Pr$  is of order unity rather than zero (Kupka 1999a,b). Hence, we included them in all the examples shown below. For the same reason, we optionally included a term in (5) that accounts for molecular dissipation effects. We use  $c_1 = 1.44$ ,  $c_2 = 1.92$ , and  $c_3 = 0.3$  where  $\beta < 0$  while  $c_3 = 0$  elsewhere (as in CD98). The local limit of (5),  $\epsilon = K^{3/2}/\Lambda$  with  $\Lambda = \alpha H_p$  and  $H_p = P/(\rho g)$ , fails to yield reasonable filling factors (Kupka 1999a) and was thus avoided.

To calculate the mean stratification we solve

$$\begin{aligned}\partial_z(P + p_t) &= -g\rho, \\ c_v \rho \partial_t T + \rho \partial_t K &= -\partial_z(F_r + F_c + F_k)\end{aligned}\quad (7)$$

where  $p_t = \overline{\rho w^2}$ . Equations (7)-(8) are taken from equation (103) of Canuto (1993) (excluding the higher order term in his equation (104)), and from equation (18c) of CD98 (for the latter,  $c_p$  was substituted to  $c_v$  to account for non-Boussinesq effects, Canuto, priv. communication). In the stationary limit, (8) yields  $F_{\text{total}} = F_r + F_c + F_k$  where the kinetic energy flux is given by  $F_k = \rho \overline{Kw}$  and the radiative flux  $F_r = -K_{\text{rad}} \partial T / \partial z$ .

Non-locality is represented by the terms  $D_f(K)$ ,  $D_f(\frac{1}{2}\overline{\theta^2})$ ,  $D_f(J)$ , and  $D_f(\frac{1}{2}\overline{w^2})$ , which require third order moments (TOMs). Using the downgradient approach (DGA) for the TOMs Kupka (1999a) found qualitatively acceptable results for the convective flux and filling factors. But quantitatively the results were not satisfactory, which corroborated the shortcomings of the DGA found in Canuto et al. (1994) and in Chan and Sofia (1996). To improve over the DGA, one must solve the dynamic equations for the TOMs (see Canuto (1992, 1993)). We investigated both the fully time dependent case which requires 6 additional differential equations and various approximations of their stationary limit which do not entail further differential equations. Here, we consider the following

“intermediate” model for the TOMs: we take the stationary limit of the dynamic equations given in Canuto (1993), but neglect the Boussinesq terms which depend on  $\beta$  (Kupka 1999b). This model is similar in robustness to the DGA, but yields significantly better results.

For the boundary conditions we impose  $T = \text{const.}$ ,  $\overline{w^2} = 0$ ,  $J = 0$ ,  $\partial K / \partial z = 0$ ,  $\partial \epsilon / \partial z = 0$ ,  $\overline{\theta^2} = 0$  at the top, while at the bottom  $\partial T / \partial z = 0$ ,  $\overline{w^2} = 0$ ,  $J = 0$ ,  $\partial K / \partial z = 0$ ,  $\partial \epsilon / \partial z = 0$ ,  $\partial \overline{\theta^2} / \partial z = 0$ . This choice permits stable numerical solutions which are consistent with the boundary conditions for the numerical simulations. The equation for the mean pressure is constrained by varying  $P$  at the top such that for a given  $T$  the resulting density stratification is consistent with mass conservation.

#### 4. Results and discussion

We have used simulation data representing two configurations: model 3J, a convection zone embedded between two strongly stable layers (Muthsam et al. 1995), and model 211P, a stably stratified layer embedded between two unstable layers (with a small stable layer at the bottom, Muthsam et al. (1999)). In both cases, radiation transports at least 80% of the input energy and  $Pr = 1$ . Each unstable layer has a thickness of  $\approx 0.5$ – $2.5 H_p$ . Model 3J has an initial adiabatic temperature contrast of 3.5 and encompasses  $4.2 H_p$  while model 211P has a contrast of 6.0 and encompasses about  $4.8 H_p$ . The numerical simulations were done for  $72 \times 50 \times 50$  grid points and have successfully been compared to higher resolution simulations ( $125 \times 100 \times 100$  grid points, Kupka and Muthsam (1999)). All simulation data shown here are statistical averages over many dozen sound crossing times.

The moment equations were solved by centered finite differences on a staggered mesh. Solutions were calculated using 72 grid points and were successfully verified by comparing them with results obtained from higher resolutions of 128 to 512 grid points. Time integration was done by the Euler forward method until a stationary state was reached after 1.2 to 2.5 thermal time scales. Verification of stationarity was done by testing whether all errors in time derivatives were formally less than  $10^{-8}$  in relative units at all grid points (i.e. far smaller than the truncation error) and by

checking the strict energy flux and mass conservation required for stationary solutions.

Fig. 1 compares the convective flux of model 211P with two convection models: the non-local model with intermediate TOM and its local counterpart (the stationary, local limit of the non-local model, see CD98). Clearly, only the non-local model successfully reproduces the simulations also in the central overshooting region which connects both convection zones of model 211P.

Fig. 2 compares the filling factor  $\sigma$ , i.e. the relative area in each layer covered by upwards flowing material, of the numerical simulations of model 3J with the filling factor computed from the non-local moment equations. For the latter we used the prescription described in CD98. As  $\sigma$  is an important topological quantity describing the inhomogeneous nature of convection, the reasonable agreement found here is a very promising result. It illustrates the importance of improving the TOMs, because the DGA used in Kupka (1999a) only indicated a correct trend, while the more complete TOM used here also provides a closer quantitative agreement. The local convection model predicts a structureless  $\sigma = 0.5$ .

Fig. 3 compares the convective flux from numerical simulations for model 3J with solutions from the moment equations using the intermediate TOM. Clearly, the latter has improved over the DGA (Kupka 1999a) both qualitatively and quantitatively. Except for the overshooting region the DGA barely differed from the local model which is shown here for comparison. Though the present convection model provides a substantial improvement, the convective flux still falls short in the middle of the convection zone of 3J and the extent of the lower overshooting zone is underestimated. This may be due to neglecting effects of radiative losses on the time scales  $\tau_\theta$  and  $\tau_{p\theta}$  as well as because several contributions to the TOMs are neglected in our “intermediate model”, or due to the Boussinesq treatment of the TOMs in (Canuto 1992, 1993), or incompleteness of equation (5).

Fig. 4 compares local length scales  $\Lambda = \alpha H_p$  as used in mixing length theory with the length scale  $\Lambda = K^{1.5}/\epsilon$  obtained with the non-local moment equations for the case of model 211P. Obviously, there is no  $\alpha$  that brings the length scales in agreement. Thus, there is no alternative but to solve at least the full equations (5)–(6) to obtain  $\epsilon$ .

In conclusion, we have found not only qualitative, but also quantitative agreement between numerical simulations and the non-local moment equations, provided one avoids the DGA for the TOMs and employs the stationary solution of their dynamic equations as suggested in appendix B of (Canuto 1993) and neglects Boussinesq type factors (involving  $\beta$ ). Moreover, we have found the mean values for  $T$ ,  $P$ , and  $\rho$  to be accurate to within 2% in comparison with 4% found for local models with optimized mixing length parameter. Convective and radiative flux are typically accurate to within 20%. The improvements are largest in regions of weak convection and in stably stratified layers. We have used the closure constants suggested in Canuto (1993) and CD98 except for  $c_{11}$ , which was increased from 0.2 to 0.5, because this improved the results for the planetary boundary layer (Canuto priv. communication) and enhanced the numerical stability. We have not found the DGA to be able to yield a similar agreement for both 3J and 211P even when tuning the closure constants individually for 3J and 211P. The situation is even worse for the local model. In Kupka (1999b) we will deal with this problem in detail. Finally, while each 3D simulation took between several days and several weeks on a modern workstation to obtain a thermally relaxed solution, the moment equations took a couple of minutes to half an hour. This holds true already for a low numerical resolution and for an explicit time integration method.

The results found here are promising for the application of the non-local convection model in particular to envelope convection zones of A and F stars, because they feature a similar range of convective efficiency, thickness in terms of  $H_p$ , and interaction among neighbouring convection zones. For A and F stars the thermal structure is not known in advance. Hence, thermal relaxation and thus the gain on speed in comparison with numerical simulations is essential. The computational savings are very attractive also for problems studied on hydrostatic time scales, whenever the full information provided by a simulation is not needed. Improvements of the convection model studied are possible, nevertheless it is a more promising basis for asteroseismological studies of pulsating A and F stars ( $\delta$  Sct,  $\gamma$  Dor, roAp, etc.) than local convection models and also a new

basis for related work such as diffusion calculations. Detailed results for a broader range of physical parameters, in particular for efficient convection and deeper convection zones, and thus of high importance also for other types of stars, have yet to corroborate this study.

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Fig. 1.— Convective flux (model 211P) for the non-local convection model and for its local limit (mixing length  $\Lambda = H_p$ ), and a numerical simulation.

Fig. 2.— Filling factor  $\sigma$  (model 3J) for the non-local convection model and a numerical simulation. For the local convection model  $\sigma = 0.5$ .

Fig. 3.— Convective flux (model 3J) for the non-local convection model and for its local limit (mixing length  $\Lambda = H_p$ ), and a numerical simulation.

Fig. 4.— Length scale  $\Lambda = K^{1.5}/\epsilon$  (model 211P) from the non-local convection model compared with  $\Lambda = \alpha H_p$  for  $\alpha = 0.7, 2.0$ .









